ARTICLE IN PRESS

Information Sciences xxx (2010) xxx–xxx

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Inter-particle communication and search-dynamics of *lbest* particle swarm optimizers: An analysis

Sayan Ghosh [a], Swagatam Das [a,*], Debarati Kundu [a], Kaushik Suresh [a], Ajith Abraham [b]

[a] Department of Electronics and Telecommunication Engineering, Jadavpur University, Kolkata, India

[b] Machine Intelligence Research Labs (MIR Labs), Scientific Network for Innovation and Research Excellence (SNIRE),Washington State 98071, USA

ARTICLE INFO

Article history:
Available online xxxx

Keywords:
Particle swarm optimization
Neighborhood topologies
Probability distribution
Swarm intelligence
Global optimization
State-space model

ABSTRACT

Particle Swarm Optimization (PSO) is arguably one of the most popular nature-inspired algorithms for real parameter optimization at present. The existing theoretical research on PSO focuses on the issues like stability, convergence, and explosion of the swarm. However, all of them are based on the gbest (global best) communication topology, which usually is susceptible to false or premature convergence over multi-modal fitness landscapes. The present standard PSO (SPSO 2007) uses an lbest (local best) topology, where a particle is stochastically attracted not towards the best position found in the entire swarm, but towards the best position found by any particle in its topological neighborhood. This article presents a first step towards a probabilistic analysis of the particle interaction and information exchange in an lbest PSO with variable random neighborhood topology (as found in SPSO 2007). It addresses issues like the distribution of particles over neighborhoods, the probability distributions of the social and cognitive terms in lbest model, and the explorative power of the lbest PSO. It also presents a state-space model of the lbest PSO and draws important conclusions regarding the stability and convergence of the particle dynamics in the light of control theory.

1. Introduction

The concept of particle swarm, although initially introduced for simulating human social behavior, has become very popular these days as an efficient means for intelligent search and optimization. The Particle Swarm Optimization (PSO) [9,10,13,16], as it is called now, does not require any gradient information of the function to be optimized, uses only primitive mathematical operators, and is conceptually very simple. Since its inception in 1995, PSO has attracted a great deal of attention of the researchers all over the globe resulting into nearly uncountable number of variants of the basic algorithm, theoretical and empirical investigations of the dynamics of the particles, parameter selection and control, and applications of the algorithm to a wide spectrum of real world problems from diverse fields of science and engineering. For a comprehensive knowledge on the foundations, perspectives, and applications of PSO see [1,2,8,10].

The first stability analysis of the particle dynamics was due to Clerc and Kennedy in 2002 [6]. F van den Bergh undertook an independent theoretical analysis of the particle swarm dynamics in his Ph.D. thesis [27], published in the same year. In [6], Clerc and Kennedy considered a deterministic approximation of the swarm dynamics by treating the random coefficients as constants, and studied stable and limit cyclic behavior of the dynamics for the settings of appropriate values to its

* Corresponding author.
E-mail addresses: sayan88tito@gmail.com (S. Ghosh), swagatam@etce.jdvu.ac.in (S. Das), kundu.debarati@gmail.com (D. Kundu), kaushik_s1988@yahoo.com (K. Suresh), ajith.abraham@ieee.org (A. Abraham).

0020-0255/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.ins.2010.10.015

parameters. A more generalized stability analysis of particle dynamics based on Lyapunov stability theorems was undertaken by Kadirkamanathan et al. [12]. Recently Poli in [22] analyzed the characteristics of a PSO sampling distribution and explained how it changes over any number of generations, in the presence of stochasticity, during stagnation. Some other significant works towards the theoretical understanding of PSO can be found in [23,25,5,11]. However, to the best of our knowledge, all the theoretical research works including the above-mentioned studies on PSO are centered on the *gbest* PSO model, where a particle is attracted towards the single best position found in the entire swarm at any iteration. The *gbest* PSO, however, is susceptible to premature and/or false convergence over the multi-modal fitness landscapes [4,7]. The current standard PSO (SPSO 2007) [3,4], obtainable from the Particle Swarm Central (http://www.particleswarm.info/) uses an *lbest* topology where each particle is stochastically attracted to the best solution that any particle in its topological *neighborhood* has found.

As will be evident from the following sections, the statistical properties of an *lbest* PSO will depend on the particular neighborhood topology used for selecting the global best for each particle. Over the years several different topologies have been proposed by the researchers. Since in this work we intended to provide a mathematical analysis (for the first time, to the best of our knowledge) of the *lbest* PSO, and it is quite impossible to take into account all the possible topologies, we selected the variable neighborhood topology since it is integrated in the current Standard PSO (SPSO 2007) model. Also, according to [7,21], an advantage of the variable random topology over other fixed topologies like wheel, star etc. is greater robustness. For a given problem, we can always find that a given fixed topology works better. But when the performance is averaged (in terms of success rate) on a set of non-biased various problems (not too similar), an adaptive variable topology is usually better [7], than a fixed topology. Suppose we are handling a pure black box optimization, i.e. we know nothing about the problem, but its search space (and sometimes some constraints too) and how to evaluate the fitness on any point of this search space. In this context, with a fixed neighborhood topology there is a "chance" that the result is very bad. With a variable one, the probability of such a failure is smaller. Although our analysis takes into account the variable topology only, as we will see, the framework can also be extended to other fixed topologies as well, with some modifications.

In this work, we provide a simple probabilistic analysis of the information exchange among the particles in *lbest* PSO using the variable random topology model of SPSO 2007. We also investigate the probability distributions of the social and cognitive terms over iterations. The analysis provides important insights into the process of choosing the informants by a particle in variable random neighborhood. It also focuses on the relative explorative powers of the *lbest* and *gbest* PSOs. Finally it derives a simple state-space model of the dynamics of a particle in *lbest* PSO and draws a few important conclusions regarding the stability and asymptotic convergence of the particle. The analysis undertaken in this paper is the first of its kind and will provide a basis for the future theoretical investigation of the internal search mechanisms of *lbest* PSO with various other topologies for improving the performance of the algorithm.

2. The particle swarm optimization algorithm

2.1. The classical PSO

The classical PSO [10,16] starts with the random initialization of a population of candidate solutions (particles) over the fitness landscape. However, unlike other evolutionary computing techniques, PSO uses no direct recombination of genetic material between individuals during the search. Rather it works depending on the social behavior of the particles in the swarm. Therefore, it finds the global best solution by simply adjusting the trajectory of each individual towards its own best position and toward the best neighboring particle at each time-step (generation).

In a D-dimensional search space, the position vector of the ith particle is given by $\vec{X}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,D})$ and velocity of the ith particle is given by $\vec{V}_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,D})$. Positions and velocities are adjusted and the objective function to be optimized i.e. $f(\vec{X}_i)$ is evaluated with the new positional coordinates at each time-step. The velocity and position update equations for the dth dimension of the ith particle in the swarm may be represented as:

$$v_{i,d,t} = \omega * v_{i,d,t-1} + C_1 * rand_1 * \left(p^l_{i,d,t-1} - x_{i,d,t-1}\right) + C_2 * rand_2 * \left(p^g_{i,d,t-1} - x_{i,d,t-1}\right), \tag{1}$$

$$x_{i,d,t} = x_{i,d,t-1} + v_{i,d,t}, \tag{2}$$

where $rand_1$ and $rand_2$ are random positive numbers uniformly distributed in $(0,1)$ and are drawn anew for each dimension of each particle. $\vec{p}^l_i$ is the personal best solution found so far by an individual particle while $\vec{p}^g_i$ represents the best particle in a neighborhood of the ith particle, for *lbest* PSO model. Note that in PSO, a neighborhood is defined for each individual particle as the subset of particles which it is able to communicate with. The *gbest* PSO may be regarded as a special case of the *lbest* model where the entire swarm acts as the neighborhood of any particle and $\vec{p}^g_i$ simply becomes the globally best position found so far by all the particles in the population. In *lbest* PSO, if at any iteration a particle is the best in its neighborhood, then the velocity update formula for this particle will be:

$$v_{i,d,t} = \omega * v_{i,d,t-1} + C_1 * rand_1 * \left(p^l_{i,d} - x_{i,d,t-1}\right). \tag{3}$$

The term $C_1 * rand_1 * \left(p^l_{i,d,t-1} - x_{i,d,t-1}\right)$ in the velocity updating formula of (1) represents a linear stochastic attraction of the particle towards the best position found so far by itself. In literature researchers have called this component as "memory,"

Please cite this article in press as: S. Ghosh et al., Inter-particle communication and search-dynamics of *lbest* particle swarm optimizers: An analysis, Inform. Sci. (2010), doi:10.1016/j.ins.2010.10.015

"self-confidence," "cognitive part" or "remembrance." The last term of the same formula i.e. $C_2 * rand_2 * \left(p^g_{i,d,t-1} - x_{i,d,t-1}\right)$ is interpreted as the 'social part', which represents how an individual particle is influenced by the other members of its society. C_1 and C_2 are called acceleration coefficients and they determine the relative influences of the cognitive and social parts on the velocity of the particle. The particle's velocity may be optionally clamped to a maximum value $\vec{V}_{\max} = [v_{\max,1}, v_{\max,2}, \ldots, v_{\max,D}]^T$. If in dth-dimension, $|v_{i,d}|$ exceeds $v_{\max,d}$ specified by the user, then the velocity of that dimension is assigned to $sign(v_{i,d})^* v_{\max,d}$, where $sign(x)$ is the triple-valued signum function.

2.2. Topological variants of the classical PSO

The basic PSO algorithm used in most of the existing papers implicitly uses a fully connected neighborhood topology (or *gbest*). Every particle is a neighbor of every other particle. Hence all particles are stochastically attracted towards the best solution found so far by any member of the swarm. Here each particle has access to the information of all other members in the community.

However, local neighborhood models (or *lbest*) have also been proposed for PSO long ago, where each particle has access to the information corresponding to its immediate neighbors, according to a certain swarm topology. The two most common topologies are the ring topology, in which each particle is connected with two neighbors and the wheel topology (typical for highly centralized business organizations), in which the individuals are isolated from one another and all the information is communicated to a focal individual. Kennedy and Mendes [14,15] evaluated a number of topologies presented as well as the case of random neighbors. In [19,20] Mendes et al. suggest that the *gbest* version converges fast but can be trapped in a local optimum very often, while the *lbest* network has more chances to find an optimal solution, although with slower convergence. As shown by the authors [19], the Von Neumann topology may perform better than other topologies including the *gbest* version.

Nevertheless, selecting the most efficient neighborhood structure, in general, depends on the type of problem. One structure may perform more effectively for certain types of problems, yet have a worse performance for other problems. The current standard PSO (SPSO'07 [7]) uses an *lbest* network with variable random neighborhood model and our present analysis will mostly be based on this model only. It is described in more details in the next subsection.

2.3. The variable random topology

The variable neighborhood topology is described in Maurice Clerc's book on PSO [7] and it can be seen now as a particular case of the stochastic star of the work of Miranda et al. [21]. In this topology, there is no centralized concept of a global best. The particles select each other as informants, and out of these informants, one particle may be selected as the target particle's global best. The topology is highly dependent on a threshold probability p, which is constant for all particles in the swarm. Each particle assigns an uniformly distributed random value (between 0 and 1) to every other particle in the swarm. Then it checks how many of these particles have values less than the threshold p. The particles having values less than p are chosen as informants, implying that the target particle will attempt to select its global best from these particles. The best particle among the informants is chosen as the global best for the target particle. If the particle's own fitness is better than the best informant, the particle simply takes its own locally best position into account.

3. Analysis of the inter-particle communication in *lbest* PSO (SPSO 2007)

3.1. Probabilities of selection of informants by a particle

Without loss of generality, in the analysis that follows, we assume that the particles are arranged in an ascending order of their locally best fitness. From now on, when we refer to the ith particle, we mean the ith ranked particle. In the following theorems we shall derive the probabilities that the ith particle selects the jth particle i.e. the jth ranked particle is selected as the globally best position by the ith ranked particle. We shall show that a particle cannot select particles inferior to it.

Theorem 1. *If P_{ij} denotes the probability that the ith ranked particle selects the jth ranked particle as its global best where $i < j$ then $P_{ij} = 0$.*

Proof. The ith particle compares its own fitness with the best fitness of the k-selected informants. If the best particle among k members (here, the jth particle) is worse than the fitness of the ith particle then it cannot be selected. Thus the probability that the ith ranked particle selects the jth ranked particle as its global best becomes zero. □

Lemma 1. *If n denotes the swarm size then the probability that the ith ranked particle chooses itself (i.e. uses Eq. (3)) when the number of chosen informants is k, is given by:*

$$P_{ii,k} = \frac{{}^{n-i}C_k}{{}^{n-1}C_k}. \tag{4}$$

Proof. The ith particle chooses itself if it cannot find a particle superior to it among the chosen k members. Thus the chosen k particles consist only of particles inferior to it. Since there are $n - i$ particles inferior to it, the number of such possible combinations is ${}^{n-i}C_k$. The total number of all possible combinations is given by ${}^{n-1}C_k$. Hence the probability that the particle chooses itself is given by $P_{ii,k} = \frac{{}^{n-i}C_k}{{}^{n-1}C_k}$. (Proved) □

Lemma 2. *If $P_{ij,k}$ denotes the probability that the ith ranked particle selects the jth ranked particle as its global best where $i > j$ and exactly k informants are chosen, then*

$$P_{ij,k} = \frac{{}^{n-1-j}C_{k-1}}{{}^{n-1}C_k}. \tag{5}$$

Proof. The jth particle can be selected only if it is superior to all other particles from the chosen k particles. There are $n-j$ particles inferior to the jth particle and there are $n-j-1$ particles inferior when we exclude the selecting particle itself. We are effectively selecting $k-1$ particles, since the jth particle is already present among the chosen k particles. The total number of such combinations is given by ${}^{n-1-j}C_{k-1}$. The total number of all possible combinations is given by ${}^{n-1}C_k$. The probability is hence

$$P_{ij,k} = \frac{{}^{n-1-j}C_{k-1}}{{}^{n-1}C_k}. \quad \text{(Proved)} \qquad \square$$

An important observation follows. First, when the jth ranked particle is selected by inferior particles, the selection probability is the same for all particles inferior to the jth particle. The result of Lemma 3 shows us that $P_{ij,k} = \frac{{}^{n-1-j}C_{k-1}}{{}^{n-1}C_k}$ is dependent only on j, n, and k.

In the following theorems, we find the respective probabilities that the i th particle follows the j th particle's locally best position, and that it follows its own locally best position. The results depend to a large extent on the value of p, the probability with which each particle is selected as an informant.

Theorem 2. *The probability that the ith particle follows the locally best position of the jth particle is given by:*

$$P_{ij} = p.(1-p)^{j-1}. \tag{6}$$

Proof. We first derive the probability with which exactly k informants are selected. Out of n-1 particles, there are ${}^{n-1}C_k$ ways in which k particles can be selected as informants. For each combination, the probability that k particles are chosen and n-1-k particles are not chosen as informants is given by $p^k(1-p)^{n-1-k}$. Hence the total probability that exactly k particles are chosen as informants is given by $P_k = {}^{n-1}C_k p^k (1-p)^{n-1-k}$. When exactly k informants are chosen, the probability that the ith particle follows the locally best position of the jth particle is given by $P_{ijk} = \frac{{}^{n-1-j}C_{k-1}}{{}^{n-1}C_k}$ (from Lemma 2).

We can find the probability that the ith particle follows the jth particle by summing over the entire range of k from 0 to n-1 as follows:

$$P_{ij} = \sum_{k=0}^{k=n-1} P_{ijk}.P_k = \sum_{k=0}^{k=n-1} {}^{n-1}C_k p^k (1-p)^{n-1-k} \frac{{}^{n-1-j}C_{k-1}}{{}^{n-1}C_k} = \sum_{k=0}^{k=n-1} {}^{n-1-j}C_{k-1} p^k (1-p)^{n-1-k}.$$

We substitute $\lambda = k - 1$ in the above expression to obtain:

$$P_{ij} = \sum_{\lambda=0}^{\lambda=n-2} {}^{n-1-j}C_\lambda p^{\lambda+1} (1-p)^{n-\lambda-2} = p(1-p)^{j-1} \sum_{\lambda=0}^{\lambda=n-2} {}^{n-1-j}C_\lambda p^\lambda (1-p)^{n-1-j-\lambda}.$$

The lower limit of k is zero, and so the lower limit of λ should be -1. However the value of ${}^{n-1-j}C_\lambda$ becomes zero for $\lambda = -1$, so we neglect the lower limit, and begin our summation from $\lambda = 0$. Again, j is a rank, so the inequality $1 \leqslant j \leqslant n$ holds. Hence we have $n - 1 - j \leqslant n - 2$. Further ${}^{n-1-j}C_\lambda = 0$ for $\lambda > n - 1 - j$. Thus we can shift the upper limit of summation to $\lambda = n - 1 - j$. The expression for P_{ij} is now given by:

$$P_{ij} = p(1-p)^{j-1} \sum_{\lambda=0}^{\lambda=n-1-j} {}^{n-1-j}C_\lambda p^\lambda (1-p)^{n-1-j-\lambda} = p(1-p)^{j-1}. \quad \text{(Proved)} \qquad \square$$

We arrive at the final expression through application of the binomial theorem.

Theorem 3. *The probability that the ith particle follows its own locally best position is given by:*

$$P_{ii} = (1-p)^{i-1}. \tag{7}$$

Proof. Proceeding in a similar manner to the proof of Theorem 2, we first derive the probability that exactly k informants are selected. Out of *n-1* particles, there are ${}^{n-1}C_k$ ways in which k particles can be selected as informants. For each combination, the probability that k particles are chosen and *n-1-k* particles are not chosen as informants is given by $p^k(1-p)^{n-1-k}$. Hence the total probability that exactly k particles are chosen as informants is given by $P_k = {}^{n-1}C_k p^k(1-p)^{n-1-k}$. When exactly k informants are chosen, the probability that the ith particle follows its own locally best position is given by $P_{ii,k} = \frac{{}^{n-i}C_k}{{}^{n-1}C_k}$.

We can find the probability that the ith particle follows itself by summing over the entire range of k from 0 to $n-1$ as follows:

$$P_{ii} = \sum_{k=0}^{k=n-1} P_k.P_{ii,k} = \sum_{k=0}^{k=n-1} {}^{n-1}C_k p^k(1-p)^{n-1-k}\frac{{}^{n-i}C_k}{{}^{n-1}C_k} = \sum_{k=0}^{k=n-1} {}^{n-i}C_k p^k(1-p)^{n-1-k} = (1-p)^{i-1}\sum_{k=0}^{k=n-1} {}^{n-i}C_k p^k(1-p)^{n-i-k}$$

$$= (1-p)^{i-1}\sum_{k=0}^{k=n-i} {}^{n-i}C_k p^k(1-p)^{n-i-k} = (1-p)^{i-1}. \quad \text{(Proved)} \qquad \square$$

We have shifted the upper limit of λ to $n-i$ in a manner similar to the proof of Theorem 2. Here also, we use the binomial theorem to arrive at the final expression.

The results of the theorem are highly dependent on the value of p. When $p = 1$, the algorithm corresponds to the classical *gbest* PSO, in which every particle of the swarm (excluding itself) is chosen as an informant for selection of the global best. The probability P_{ij} evaluates to 0 when $j \neq 1$ and it evaluates to 1 when $j = 1$. Thus all particles of the swarm follow the globally best position. The probability $P_{ii} = 0$ when $i \neq 1$ which implies that no particle (except the globally best one) can use Eq. (3) for velocity update. When $p = 0$, the particles cease to interact with one another, with every particle following its own locally best position. The plots in Fig. 1 show the probabilities P_{ii} and P_{ij} as functions of p.

3.2. Probability distributions

We shall now proceed with a formal analysis of the velocity update equation following the selection topology of the local best PSO. Assuming that the entire particle swarm system is known at an earlier instant of time $t-1$, we formulate the probability distributions of various terms in the update equation and contrast them with the classical PSO.

Finally we prove that in the case of the local best PSO, a particle is able to cover a greater search space area when compared to the classical PSO, leading to greater explorative power. We assume, initially that our analysis is restricted to one-dimensional space. The position update equation used in the classical PSO in scalar form is given by:

$$x_i(t) = x_i(t-1) + \omega v_i(t-1) + \varphi_1(p_i^l - x_i(t-1)) + \varphi_2(p^g - x_i(t-1)). \tag{8}$$

The position update equation used in the local best PSO is given by:

$$x_i(t) = x_i(t-1) + \omega v_i(t-1) + \varphi_1(p_i^l - x_i(t-1)) + \varphi_2(p_i^g - x_i(t-1)), \tag{9}$$

when the *i*-particle does not select itself, and

$$x_i(t) = x_i(t-1) + \omega v_i(t-1) + \varphi_1(p_i^l - x_i(t-1)), \tag{10}$$

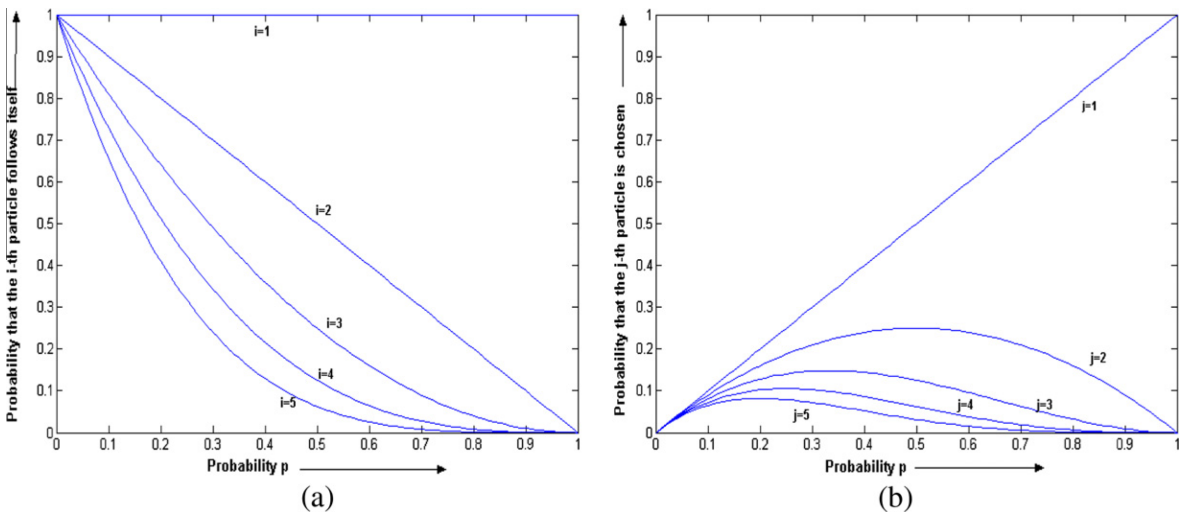


Fig. 1. (a) Variation of P_{ii} with p for different values of i and (b) Variation of P_{ij} with p for different values of j.

when the i th particle selects itself. $x_i(t-1)$ and $x_i(t)$ are the positions of the ith ranked particle at time instant $t-1$ and time instant t. $v_i(t-1)$ is velocity of the ith particle at time instant $t-1$. p_i^l is the locally best position of the ith particle at time instant $t-1$ respectively. p^g is the globally best position found by the entire swarm. p_i^g is the globally best position found by the ith particle after application of the selection rules of the *lbest* PSO. Note that $\varphi_1 = C_1{}^*rand_1$ and $\varphi_2 = C_2{}^*rand_2$ as per Eq. (1) and ω is a real constant between 0 and 1.

The velocities and positions of a particle at an instant t depend not only on their values at the instant $t-1$, but also on the random numbers φ_1 and φ_2 and on the topology for selecting the global best. Hence we begin our analysis on the assumption that the entire particle swarm system at the instant $t-1$ is completely known to us, i.e. the velocities, locally best positions and positions at the instant $t-1$ are considered as deterministic in our analysis. This enables us to build a clear picture of the extent to which a particle may be perturbed from its known original position at $t-1$ to form its new position at t. We shall eventually obtain the expressions for the total bound of perturbation, and proceed to show that for the local best PSO the particle is perturbed to a greater extent in comparison with the classical PSO. Due to a greater degree of perturbation, the particle is capable of searching more distant zones on the fitness landscape.

3.2.1. Probability distributions of various terms in the position update equation

3.2.1.1. The inertial term. This term is common to both the local best PSO and the classical PSO. The inertial term variable W_i for the ith ranked particle is defined as $W_i = \omega \cdot v_i(t-1)$, where ω is a constant and $v_i(t-1)$ is a deterministic quantity. Thus the inertial term is not a random variable, since it is the product of a known constant and a deterministic quantity. The probability distribution of the inertial term hence does not exist.

3.2.1.2. The cognitive term. We can define the term C_i for the ith ranked particle by the following expression:

$$C_i = \phi_1\left(p_i^l - x_i(t-1)\right) = \phi_1 D_{ii}, \tag{11}$$

ϕ_1 is a random number defined uniformly in the range $(0,1)$. The probability distribution of C_i, assuming that D_{ii} is deterministic, is given by:

$$p_{C_i}(x) = \frac{1}{D_{ii}}, \quad \text{for } 0 \leqslant x \leqslant D_{ii} = 0, \quad \text{for } x > D_{ii} \quad \text{or} \quad x < 0, \tag{12}$$

when $D_{ii} > 0$, and,

$$p_{C_i}(x) = \frac{1}{|D_{ii}|}, \quad \text{for } D_{ii} \leqslant x \leqslant 0 = 0 \quad \text{for } x < D_{ii} \text{ or } \quad x > 0, \tag{13}$$

when $D_{ii} < 0$.

3.2.1.3. The social term. We can define the term S_i for the local best PSO as:

$$S_i = \phi_2(p_i^g - x_i(t-1)), \tag{14}$$

when the ith ranked particle selects a superior particle, and $S_i = 0$ when the ith ranked particle selects itself. Now the ith particle selects itself with probability P_{ii} and the jth particle with probability P_{ij}. Obviously $P_{ij} = 0$ for $i < j$. Thus we can expand the definition of S_i for the local best PSO as follows:

$$\begin{aligned}
&S_i = 0 \text{ with probability } P_{ii}\\
&S_i = \phi_2(p_1^l - x_i(t-1)) = \phi_2 D_{i1} \text{ with probability } P_{i1}\\
&S_i = \phi_2(p_2^l - x_i(t-1)) = \phi_2 D_{i2} \text{ with probability } P_{i2}\\
&\quad\cdot \quad \cdot \quad \cdot\\
&\quad\cdot \quad \cdot \quad \cdot\\
&\quad\cdot \quad \cdot \quad \cdot\\
&S_i = \phi_2(p_{i-1}^l - x_i(t-1)) = \phi_2 D_{i,i-1} \text{ with probability } P_{i,i-1}.
\end{aligned}$$

The probability distribution of S_i can be written concisely below:

$$p_{S_i}(x) = P_{ii}\delta(x) + \sum_{j=1}^{i-1} P_{ij}G_{ij}(x), \tag{15}$$

where

$$G_{ij}(x) = \frac{1}{|D_{ij}|} \quad \text{for } 0 \leqslant x \leqslant D_{ij} = 0 \quad \text{for } x < 0 \quad \text{and} \quad x > D_{ij},$$

when $D_{ij} > 0$

$$G_{ij}(x) = \frac{1}{|D_{ij}|} \quad \text{for } D_{ij} \leqslant x \leqslant 0 = 0 \quad \text{for } x > 0 \quad \text{and} \quad x < D_{ij},$$

when $D_{ij} < 0$.

Hence the probability distribution of S_i consists of a single impulse at the origin, and several gate functions on both sides of the origin, which add up to form a step function. We can define the social term S_i for the classical PSO as $S_i = \phi_2(p^g - x_i(t-1))$ where p^g is the globally best position of the entire swarm which is deterministic and is utilized by all particles in the swarm.

The probability distribution of S_i can be written as:

$$p_{S_i}(x) = \frac{1}{|D_{i1}|}, \quad \text{for } 0 \leqslant x \leqslant D_{i1} = 0, \quad \text{for } x > D_{i1} \quad \text{and} \quad x < 0, \tag{16}$$

when $D_{i1} > 0$

$$p_{S_i}(x) = \frac{1}{|D_{i1}|}, \quad \text{for } D_{i1} \leqslant x \leqslant 0 = 0, \quad \text{for } x < D_{i1} \quad \text{and} \quad x > 0, \tag{17}$$

when $D_{i1} < 0$.

The probability distribution contains a single gate pulse, which may lie to the left or right of the origin, as opposed to the distribution for the local best PSO which consists of an impulse function, as well as several gate pulses, depending on how many superior particles are referenced in the update equation.

The results derived in the previous section can be applied to a sample case, which defines the initial position of the swarm at time instant $t-1$. We can obtain the distribution of the cognitive and social terms for a particular particle. The swarm size is assumed to be 5 particles and the value of p is assumed to be 0.4. For example, we are interested in obtaining the distribution for the 4th ranked particle. Table 1 shows the configuration of the 4th ranked particle at time instant $t-1$. Fig. 2(a) shows the distribution of the cognitive term for the 4th ranked particle. This distribution is common to both topologies. Fig. 2(b) and (c) show the distribution of the social terms for the 4th particle in the case of the classical PSO and the local best PSO respectively.

3.3. Lower and upper bounds

We can write the expression for the position $x_i(t)$ for the ith particle at the tth time instant as:

$$x_i(t) = x_i(t-1) + W_i + C_i + S_i, \tag{18}$$

where W_i represents the inertial term, C_i represents the cognitive term and S_i represents the social term. Since $x_i(t-1)$ and W_i are deterministic, the only random variables are C_i and S_i. We have already derived the expressions for the probability distributions of these random variables. The probability distribution of the particle's position at all instants of time is bounded, and we are primarily interested in determining the upper and lower bounds of this distribution. The range of the distribution is a measure of the explorative capability of the particle, and it is the difference between the upper and lower bounds of the distribution. In the above expression for the position, the term C_i have same distribution for the classical PSO and the local best PSO, while S_i is topology dependent and hence is different for the two topologies. We can express the lower and upper bounds of a particle's position as:

$$LB[x_i(t)] = x_i(t-1) + W_i + LB[C_i] + LB[S_i] \tag{19}$$

$$UB[x_i(t)] = x_i(t-1) + W_i + UB[C_i] + UB[S_i], \tag{20}$$

where $LB[R]$ and $UB[R]$ denote the lower and upper bounds respectively, of the random variable R.

Table 1
Configuration of the 4th ranked particle at a time instant $t-1$.

Position $x_4(t-1)$	5.234
Velocity $v_4(t-1)$	−2.176
$D_{41} = p_1^l - x_4(t-1)$	8.274
$D_{42} = p_2^l - x_4(t-1)$	2.543
$D_{43} = p_3^l - x_4(t-1)$	−6.780
$D_{44} = p_4^l - x_4(t-1)$	−3.869
$D_{45} = p_5^l - x_4(t-1)$	4.218

Please cite this article in press as: S. Ghosh et al., Inter-particle communication and search-dynamics of *lbest* particle swarm optimizers: An analysis, Inform. Sci. (2010), doi:10.1016/j.ins.2010.10.015

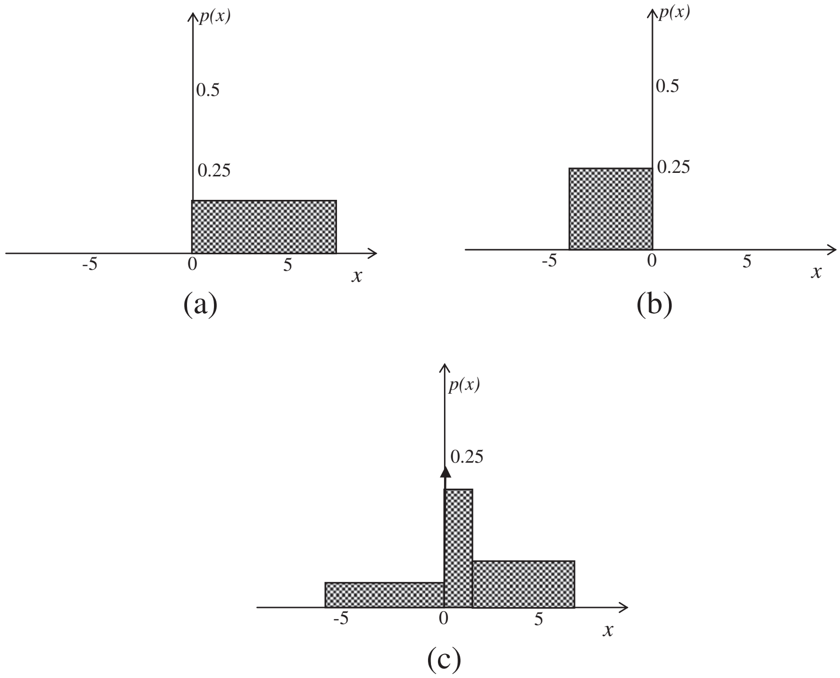


Fig. 2. The probability distributions of the (a) cognitive term in *lbest* and *gbest* PSO (b) social term in classical *gbest* PSO, and (c) social term in *lbest* PSO for the update equation of the 4th ranked particle.

Definition. For the *i*th ranked particle, the social search space range $\sum_{i,\tau}$ of the social term S_i is defined to be the difference between the upper bound and the lower bound of S_i in the topology τ. Hence $\sum_{i,\tau} = UB[S_i] - LB[S_i]$ for topology τ. In case of the topology τ_1 corresponding to the classical PSO, the lower and upper bounds of S_i are:

$$\begin{aligned} LB[S_i] &= D_{i1} \quad \text{if } D_{i1} < 0, \\ &= 0 \quad \text{if } D_{i1} \geqslant 0, \\ UB[S_i] &= D_{i1} \quad \text{if } D_{i1} > 0, \\ &= 0 \quad \text{if } D_{i1} \leqslant 0. \end{aligned}$$

In case of the topology τ_2 corresponding to the local best PSO, we initially define two sets ρ_i and ξ_i as:

$$\rho_i = \{D_{ij} | j < i \quad \text{and} \quad D_{ij} < 0, i, j \in \mathrm{N}\} \quad \text{and} \quad \xi_i = \{D_{ij} | j < i \quad \text{and} \quad D_{ij} \geqslant 0; i, j \in \mathrm{N}\}.$$

Then the upper and lower bounds of S_i are:

$$\begin{aligned} LB[S_i] &= D_{i1} \quad \text{if } D_{i1} < 0, \\ &= 0 \quad \text{if } D_{i1} \geqslant 0, \\ UB[S_i] &= D_{i1} \quad \text{if } D_{i1} > 0, \\ &= 0 \quad \text{if } D_{i1} \leqslant 0. \end{aligned}$$

In case of the topology τ_2 corresponding to the local best PSO, we initially define two sets ρ_i and ξ_i as:
$\rho_i = \{D_{ij} | j < i$ and $D_{ij} < 0, i, j \in \mathrm{N}\}$ and $\xi_i = \{D_{ij} | j < i$ and $D_{ij} \geqslant 0; i, j \in \mathrm{N}\}$. Then the upper and lower bounds of S_i are:

$$\begin{aligned} LB[S_i] &= 0 \quad \text{if } \rho_i = \phi \qquad & UB[S_i] &= 0 \quad \text{if } \xi_i = \phi \\ &= \min(\rho_i) \quad \text{if } \rho_i \neq \phi & &= \max(\xi_i) \quad \text{if } \xi_i \neq \phi \end{aligned}$$

Lemma 3. *For the ith particle if $\sum_{i,1}$ denotes the search space range of the social term S_i for topology τ_1 corresponding to the classical PSO and $\sum_{i,2}$ denotes the search space range of S_i for topology τ_2 corresponding to the local best PSO then $\sum_{i,2} \geqslant \sum_{i,1}$ for $1 \leqslant i \leqslant n$.*

Proof. We can write the expressions for the search space ranges corresponding to topologies τ_1 and τ_2 as:

$$\sum_{i,2} = \max(\xi_i) - \min(\rho_i) \quad \text{when } \xi_i \neq \phi \text{ and } \rho_i \neq \phi$$

$$\sum_{i,1} = |D_{i1}| \qquad \begin{aligned} &= \max(\xi_i) && \text{when } \xi_i \neq \phi \text{ and } \rho_i = \phi \\ &= -\min(\rho_i) && \text{when } \xi_i = \phi \text{ and } \rho_i \neq \phi \\ &= 0 && \text{when } \xi_i = \phi \text{ and } \rho_i = \phi \end{aligned}$$

1. For the case when $D_{i1} < 0$, $D_{i1} \in \rho_i$ and hence, $\rho_i \neq \phi$ and $\sum_{i,2} = \max(\xi_i) - \min(\rho_i)$ when $\xi_i \neq \phi$ and $\sum_{i,2} = -\min(\rho_i)$ when $\xi_i = \phi$. Then min $(\rho_i) \leqslant D_{i1}$

$$\Rightarrow -\min(\rho_i) \geqslant -D_{i1},$$
$$\Rightarrow -\min(\rho_i) \geqslant |D_{i1}||D_{i1}| = -D_{i1}.$$

   Since max $(\xi_i) > 0$ we have

$$\max(\xi_i) - \min(\rho_i) \geqslant |D_{i1}|.$$

   Hence when $D_{i1} < 0, \sum_{i,2} \geqslant \sum_{i,1}$
2. For the case when $D_{i1} \geqslant 0$, $D_{i1} \in \xi_i$ and hence $\xi_i \neq \phi$ and $\sum_{i,2} = \max(\xi_i) - \min(\rho_i)$ when $\rho_i \neq \phi$, and $\sum_{i,2} = \max(\xi_i)$ when $\rho_i = \phi$. Then we have max $(\xi_i) \geqslant D_{i1} \Rightarrow$ max $(\xi_i) \geqslant |D_{i1}|$ as $D_{i1} \geqslant 0$. Since min $(\rho_i) < 0$, we have max (ξ_i) – min $(\rho_i) \geqslant |D_{i1}|$. Hence when $D_{i1} > 0, \sum_{i,2} \geqslant \sum_{i,1}$. Hence Lemma 3 is proved. □

Definition. For the ith ranked particle, the search space range $\Gamma_{i,\tau}$ of the position $x_i(t)$ is defined to be the difference between the upper bound and the lower bound of $x_i(t)$ in the topology τ.

Hence $\Gamma_{i,\tau} = UB[x_i(t)] - LB[x_i(t)]$ for topology τ.

Theorem 4. *For the ith particle if $\Gamma_{i,1}$ denotes the search space range of the position $x_i(t)$ for topology τ_1 corresponding to the classical PSO and $\Gamma_{i,2}$ denotes the search space range of $x_i(t)$ for topology τ_2 corresponding to the local best PSO then $\Gamma_{i,2} \geqslant \Gamma_{i,1}$ for $1 \leqslant i \leqslant n$.*

Using Eqs. (19) *and* (20) *we have:*

$$\Gamma_{i,\tau} = UB[C_i] - LB[C_i] + UB[S_i] - LB[S_i],$$

where the first four terms on the right hand side are the same for both the classical PSO and the local best PSO. These four terms sum up to form the term η_i which is common to both topologies. The terms corresponding to the social dynamics are different and can be combined to form $\sum_{i,\tau}$. Thus we can write:

$$\Gamma_{i,1} = \eta_i + \sum_{i,1},$$
$$\Gamma_{i,2} = \eta_i + \sum_{i,2}.$$

Since $\sum_{i,2} \geqslant \sum_{i,1}, \Gamma_{i,2} \geqslant \Gamma_{i,1}$. Hence the theorem is proved.

The search space range defines the extent to which a particle may be perturbed due to the velocity and position update equations. Greater the search space range, greater the explorative power of the particle. From an intuitive interpretation of Theorem 4, we conclude that the variable random topology is superior to the topology used in the classical PSO in the sense that it provides a greater explorative power to the particle. This is amply illustrated in Fig. 2(b), where we observe that as an example, the social term introduces a perturbation from $x = 0$ to $x = 8$, thus resulting in a search space range of $\Gamma = 8$, while in Fig. 2(c) the corresponding perturbation in the case of the local best PSO is from $x = -6$ to $x = 8$ and results in a range of $\Gamma = 14$.

4. A state-space analysis of particle dynamics in lbest PSO

In the following analysis we model the particle velocities and positions as state variables. It is assumed that the locally best positions of all particles do not change with time, and each particle has its distinct local best position. Consequently, the ranks of the particles are static, since the ranks are based only on the locally best positions. With this assumption we find the state space representation of the system [17,18,28] and show how the use of the variable random topology leads to a change in the system matrix and hence affects the convergence rate of the expected positions and velocities. Conditions leading to the asymptotic convergence of the expectation values of position and velocity of the particle are also investigated.

Consider the velocity and position update equations for the *lbest* PSO (on a one-dimensional search space) using variable random topology in a slightly different way as:

$$v_i(t) = \omega v_i(t-1) + \varphi_1(p_i^l - x_i(t-1)) + \varphi_2(p_i^g - x_i(t-1)), \tag{21a}$$

$$x_i(t) = x_i(t-1) + \omega v_i(t-1) + \varphi_1(p_i^l - x_i(t-1)) + \varphi_2(p_i^g - x_i(t-1)), \tag{21b}$$

when the particle selects another particle as its global best, and

$$x_i(t) = x_i(t-1) + \omega v_i(t-1) + \varphi_1(p_i^l - x_i(t-1)), \tag{21c}$$

when it selects itself.

Since the ith particle selects other particles with probabilities $P_{i1}, P_{i2}, P_{i3}, \ldots. P_{i,i-1}$ and itself with probability P_{ii}, the expectation value of the position and velocities of the ith particle can be expressed as:

$$\begin{aligned} Ev_i(t) &= \omega Ev_i(t-1) + E(\phi_1)(p_i^l - Ex_i(t-1)) + E(\phi_2)\sum_{j=1}^{i-1} P_{ij}(p_j^l - Ex_i(t-1)) \Rightarrow Ev_i(t) \\ &= \omega Ev_i(t-1) - \left(E(\phi_1) + E(\phi_2)\sum_{j=1}^{j=i-1} P_{ij}\right)Ex_i(t-1) + E(\phi_1)p_i^l + E(\phi_2)\sum_{j=1}^{i-1} P_{ij}p_j^l \Rightarrow Ev_i(t) \\ &= \omega Ev_i(t-1) - \left(\frac{C_1}{2} + \frac{C_2}{2}(1-P_{ii})\right)Ex_i(t-1) + \frac{C_1}{2}p_i^l + \frac{C_2}{2}\sum_{j=1}^{i-1} P_{ij}p_j^l. \end{aligned} \tag{22}$$

Similarly, the expectation value of the position of the ith particle is given by:

$$Ex_i(t) = \omega Ev_i(t-1) + \left(1 - \frac{C_1}{2} - \frac{C_2}{2}(1-P_{ii})\right)Ex_i(t-1) + \frac{C_1}{2}p_i^l + \frac{C_2}{2}\sum_{j=1}^{i-1} P_{ij}p_j^l. \tag{23}$$

From the above relations, the state space representation of the system (for the ith particle) is given by:

$$\begin{bmatrix} Ex_i(t) \\ Ev_i(t) \end{bmatrix} = \begin{bmatrix} 1-\eta & \omega \\ -\eta & \omega \end{bmatrix}\begin{bmatrix} Ex_i(t-1) \\ Ev_i(t-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\left[\frac{C_1}{2}p_i^l + \frac{C_2}{2}\sum_{j=1}^{i-1} P_{ij}p_j^l\right]. \tag{24}$$

The A-matrix of the state space model is given by $A = \begin{bmatrix} 1-\eta & \omega \\ -\eta & \omega \end{bmatrix}$ where:

$$\eta = \left(\frac{C_1}{2} + \frac{C_2}{2}(1-P_{ii})\right) = \left(\frac{C_1}{2} + \frac{C_2}{2}\left(1-(1-p)^{i-1}\right)\right).$$

A block diagram representing the above state-space model has been shown in Fig. 3.

It is interesting to note how the inclusion of the term $(1-(1-p)^{i-1})$ affects the degree of social interaction of each particle. The particle, which has found the fittest *lbest* position, is given total autonomy and does not interact with other particles. As i increases, the degree of social interaction increases. Greater the degree of social interaction, greater is the explorative power. Hence the best particle is concerned only with exploitation, while the other particles not only explore their surroundings, but also exploit their locally best positions.

The characteristic equation of the A-matrix is given by $\lambda^2 + \lambda(\eta - 1 - \omega) + \omega = 0$. On application of the Jury and Blanchard's stability test [18] to investigate stability of this system, it is found that the following three conditions hold for the system to be stable and convergent:

$$(1)\ \eta > 0 \Rightarrow C_1 + C_2[1-(1-p)^{i-1}] > 0, \tag{25a}$$

$$(2)\ \eta < 2(1+\omega) \Rightarrow C_1 + C_2[1-(1-p)^{i-1}] < 4(1+\omega), \tag{25b}$$

$$(3)\ |\omega| < 1. \tag{25c}$$

Note that these conditions ensure that the poles of the system are confined within the unit circle in the Z-plane [18], i.e. the transient response of the system is damped and it settles down to constant values asymptotically. In all implementations of

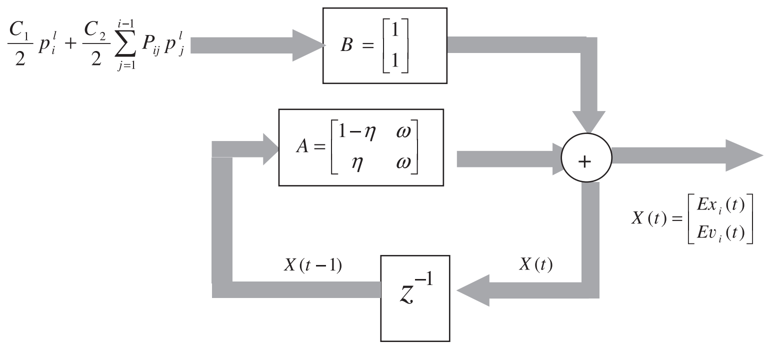
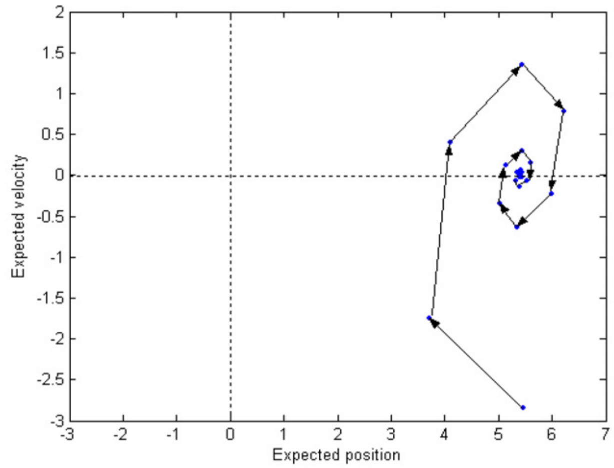
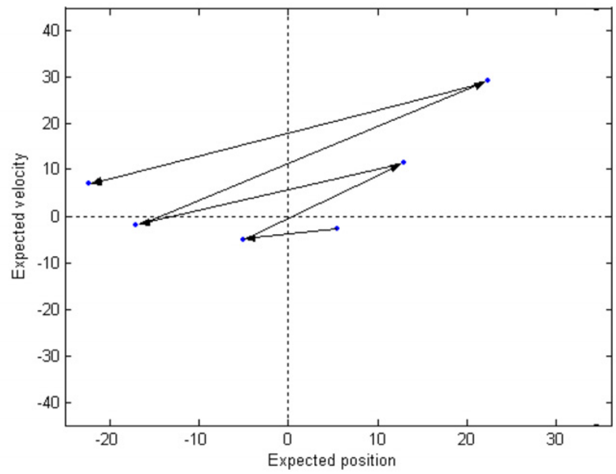


Fig. 3. Block diagram model of the state-space representation of a particle dynamics in *lbest* PSO with variable neighborhood.



(a) Stable discrete-time phase trajectory of a particle in *lbest* PSO following conditions of (25)



(b) Unstable discrete-time phase trajectory of a particle in *lbest* PSO, not obeying conditions of (25)

Fig. 4. Phase trajectory of the *i*th particle in *lbest* PSO model with variable random neighborhood topology under two different parametric settings.

the classical PSO algorithm, the inertial factor ω is between 0 and 1, hence condition (3) is satisfied. p is a probability term, hence it lies between 0 and 1. Note that in order to ensure the stability of the *gbest* PSO the cognitive and social parameters C_1 and C_2 are given appropriate values so that the condition $0 < C_1 + C_2 < 4(1 + \omega)$ is satisfied [25–27]. However, since $0 < C_1 + C_2[1 - (1-p)^{i-1}] < C_1 + C_2$ holds conditions (1) and (2) are also satisfied.

In order to support the above analysis below we provide the phase trajectories (velocity vs. position plots) of a particle in *lbest* topology with variable random neighborhood under two different settings of the control parameters ω, C_1, C_2, and p. For the first case we choose $\omega = 0.6$, $C_1 = C_2 = 2.00$, and $p = 0.5$, which obey the conditions provided in (25). As seen from Fig. 4(a) the phase trajectory spirally converges inward to a fixed point (the stable attractor) on the position axis of the particle. It ultimately converges to the point $(x_0,0)$ on the x-axis where $x_0 = c_0/\eta$ and $c_0 = \left(\frac{C_1}{2}pl_i + \frac{C_2}{2}\sum_{j=1}^{i-1}P_{ij}.pl_j\right)$. For the parameter settings we have selected to plot the state-space trajectory, $x_0$takes a value of 5.46. In the second case we choose $\omega = 0.6$, $C_1 = C_2 = 4.60$, and $p = 0.5$, a setting that does not obey the stability conditions enlisted in (25a)–(25c). Consequently we see that the discrete-time phase-trajectory of this particle is unstable and diverges away with time in a zig–zag path.

5. Conclusions

Past few years have witnessed an increasing interest of the researchers towards the theoretical understanding of different properties of the PSO algorithm. However, to the best of our knowledge, till date, all the significant research papers published in this area are concerned about the *gbest* PSO, which is prone towards premature convergence (since all the particles are attracted towards a single point of the fitness landscape) and is generally biased towards optima at the origin. The current standard PSO (SPSO 2007) uses an *lbest* PSO with variable random neighborhood topology. This work is the first of its kind that attempts to provide a theoretical foundation to the working principles of the *lbest* PSO. First it captures an inner view of the particle interaction in the variable random topology of an *lbest* PSO by deriving the probabilities with which the particles exchange information among themselves. Since the particles choose informants on basis of their locally best fitness values, we have analyzed their interaction by assigning a fitness-based rank to each of them. We have also shown that the classical topology is a special case of the variable random topology when $p = 1$. By deriving expressions for the bound of a particle's position, we have proved that the variable random topology provides a wider search space to a particle than the classical topology. Finally we present a state-space model of the swarm dynamics under certain simplifying assumptions and then deduce the necessary conditions that ensure stability and asymptotic convergence of the particle dynamics by using Jury and Blanchard's stability test, a very common technique from digital control engineering.

Further research may focus on the effect of p on the performance of the algorithm, as well as other topological variants of the variable random topology. The analysis in this paper has been specifically based on the variable random topology in which each variable chooses the best particle out of a few randomly chosen informants. It is worthy to investigate how the mathematical framework described in this article may be extended to more advanced PSO-variants [24,30,31]. We intend to undertake a similar analysis the particle interactions for a recently developed version of PSO, called Cyber Swarm Algorithms [31] that encourage the use of variable random neighborhoods and incorporate adaptive memory learning strategies derived from principles embodied in Scatter Search and Path Relinking. General steps in the analysis like probability of choosing informants, stochastic state-space model based on the expectation values of the position (which depend on the probability calculations), and the investigation of the search space bound of a particle can also be extended to other topologies as well, in which the particle chooses the locally best positions of other particles in either phenotypic or genotypic neighborhood. It will be worthy to investigate the convergence behavior of PSO-variants when applied to multi-objective optimization problems [29]. Our future works will proceed in this direction.

References

[1] A. Banks, J. Vincent, C. Anyakoha, A review of particle swarm optimization. Part I: Background and development, Natural Computing: An International Journal 6 (4) (2007) 467–484.

[2] A. Banks, J. Vincent, C. Anyakoha, A review of particle swarm optimization. Part II: Hybridisation, combinatorial, multicriteria and constrained optimization, and indicative applications, Natural Computing: An International Journal 7 (2) (2008) 109–124.

[3] D. Bratton, T. Blackwell, Understanding particle swarms through simplification: a study of recombinant PSO, in: Proceedings of the 2007 GECCO conference companion on genetic and evolutionary computation (London, United Kingdom, July 07-11, 2007), GECCO '07, ACM, New York, NY, 2007, pp. 2621–2628.

[4] D. Bratton, J. Kennedy, Defining a standard for particle swarm optimization, in: IEEE swarm intelligence symposium, 2007.

[5] E.F. Campana, G. Fasano, A. Pinto, Dynamic system analysis and initial particles position in particle swarm optimization, in: IEEE Swarm Intelligence Symposium 2006, Indianapolis 12–14 May, 2006.

[6] M. Clerc, J. Kennedy, The particle swarm-explosion, stability and convergence in a multidimensional complex space, IEEE Transactions on Evolutionary Computation 6 (2) (2002) 58–73.

[7] M. Clerc, Particle Swarm Optimization, ISTE Publications, 2008.

[8] Y. del Valle, G.K. Venayagamoorthy, S. Mohagheghi, J.C. Hernandez, R.G. Harley, Particle swarm optimization: basic concepts, variants and applications in power systems, IEEE Transactions on Evolutionary Computation 12 (2) (2008) 171–195.

[9] R.C. Eberhart, J. Kennedy, A new optimizer using particle swarm theory, in: Proceedings Sixth International Symposium Micromachine Human Science, 1, 1995, pp. 39–43.

[10] A.P. Engelbrecht, Fundamentals of Computational Swarm Intelligence, John Wiley & Sons, 2006.

[11] S. Helwig, R. Wanka, Theoretical analysis of initial particle swarm behavior, in: Proceedings of the 10th International Conference on Parallel Problem Solving from Nature, G. Rudolph et al. (Eds.), LNCS 5199, pp. 889–898, Dortmund, Germany, 2008.

[12] V. Kadirkamanathan, K. Selvarajah, P.J. Fleming, Stability analysis of the particle dynamics in particle swarm optimizer, IEEE Transactions on Evolutionary Computation 10 (3) (2006) 245–255.

[13] J. Kennedy, R.C. Eberhart : Particle swarm optimization, in: Proceedings of IEEE International conference on Neural Networks, pp. 1942–1948, 1995.

[14] J. Kennedy, R. Mendes, Population structure and particle swarm performance, in: Proceedings of the Congress on Evolutionary Computation (CEC 2002), Honolulu, HI, USA, pp. 1671–1676, 2002.

[15] J. Kennedy, Small worlds and mega-minds: effects of neighborhood topology on particle swarm performance, Proceedings of the 1999 Congress of Evolutionary Computation, 3, IEEE Press, 1999, pp. 1931–1938.

[16] J. Kennedy, R.C. Eberhart, Y. Shi, Swarm Intelligence, Morgan Kaufmann, San Francisco, CA, 2001.

[17] B.C. Kuo, Automatic Control Systems, Prentice-Hall, Englewood Cliffs, NJ, 1987.

[18] B.C. Kuo, Digital Control System, Oxford, NY, 1992.

[19] R. Mendes, J. Kennedy, J. Neves, The fully informed particle swarm: simpler, maybe better, IEEE Transactions on Evolutionary Computation 8 (3) (2004) 204–210.

[20] R. Mendes, J. Kennedy, J. Neves, Avoiding the pitfalls of local optima: how topologies can save the day, Proceedings of the 12th Conference Intelligent Systems Application to Power Systems (ISAP2003), IEEE Computer Society, 2003.

[21] V. Miranda, H. Keko, A.J. Duque, Stochastic Star Communication Topology in Evolutionary Particle Swarms (EPSO), International Journal of Computational Intelligent Research (2008).

[22] R. Poli, Mean and variance of the sampling distribution of particle swarm optimizers during stagnation, IEEE Transactions on Evolutionary Computation 13 (4) (2009) 712–721.

[23] N.R. Samal, A. Konar, S. Das, A. Abraham , A closed loop stability analysis and parameter selection of the particle swarm optimization dynamics for faster convergence, in: Proceedings of Congress on Evolutionary Computation, (CEC 2007), Singapore, pp. 1769–1776, 2007.

[24] Y. Shi, H. Liu, L. Gao, G. Zhang, Cellular particle swarm optimization, Information Sciences (2010). Available online 8 June.

[25] I.C. Trelea, The particle swarm optimization algorithm: convergence analysis and parameter selection, Information Processing Letters 85 (2003) 317–325.

[26] F. Van den Bergh, A.P. Engelbrecht, A study of particle swarm optimization particle trajectories, Information Sciences 176 (8) (2006) 937–971.

[27] Van den Bergh F., An Analysis of Particle Swarm Optimizers, Ph.D. Dissertation, University of Pretoria, South Africa, 2002.

[28] M. Vidyasagar, Nonlinear System Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1993.

Please cite this article in press as: S. Ghosh et al., Inter-particle communication and search-dynamics of *lbest* particle swarm optimizers: An analysis, Inform. Sci. (2010), doi:10.1016/j.ins.2010.10.015

[29] Y. Wang, Y. Yang, Particle swarm optimization with preference order ranking for multi-objective optimization, Information Sciences 179 (12) (2009) 1944–1959.

[30] Y. Wang, B. Li, T. Weise, J. Wang, B. Yuan, Q. Tian, Self-adaptive learning based particle swarm optimization, Information Sciences, (in press), Corrected Proof, Available online 24 July 2010, doi:10.1016/j.ins.2010.07.013.

[31] P.Y. Yin, F. Glover, M. Laguna, J.X. Zhu, Cyber swarm algorithms-improving particle swarm optimization using adaptive memory strategies, European Journal of Operational Research 201 (2) (2010) 377–389.